Addendum to “An analytical solution method for the unsteady, unbounded, incompressible three-dimensional Navier-Stokes equations in Cartesian coordinates using coordinate axis symmetry degeneracy”

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Abstract

This document provides some additional derivations and clarifications as an extension of the discussion of the derivation of the uniaxial solution found in the last section of the article “An analytical solution method for the unsteady, unbounded, incompressible three-dimensional Navier-Stoke equations in Cartesian coordinates,” available at http://resolver.caltech.edu/CaltechAUTHORS:20111225-210446344. While this material may provide valuable additional information about the Navier-Stokes solutions, no attempt has been made to develop the contents as a stand-alone document. The information contained herein makes significant reference to the major work, and should be read in the same manner as supporting information for an article in a major journal would be.

Addendum 1: Linear combination of solutions

In developing additional solutions to a problem, a common procedure is to determine under what conditions linear combinations of solutions also lead to solutions. To examine linear combinations of the solutions described in the main article, consider a velocity profile that is a linear combination of two known solutions to the Navier-Stokes equations, $\mathbf{v_1}$ and $\mathbf{v_2}$, such that:

$$\mathbf{u}(x,y,z,t) = \mathbf{v_1}(x,y,z,t) + \mathbf{v_2}(x,y,z,t)$$

Inserting this combination solution into Equations 1a of the main paper gives:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\mathbf{P} + \eta\nabla^2\mathbf{u}$$

which, expanded, gives:

$$\rho\frac{\partial \mathbf{v_1}}{\partial t} + \rho\frac{\partial \mathbf{v_2}}{\partial t} + \rho\mathbf{v_1}\cdot\nabla\mathbf{v_1} + \rho\mathbf{v_1}\cdot\nabla\mathbf{v_2} + \mathbf{v_2}\cdot\nabla\mathbf{v_1} + \mathbf{v_2}\cdot\nabla\mathbf{v_2} = -\nabla\mathbf{P} + \eta\nabla^2\mathbf{v_1} + \eta\nabla^2\mathbf{v_2}$$

and Equation 1b from the main paper is trivially solved. Because $\mathbf{v_1}$ and $\mathbf{v_2}$ are solutions to Equation 1a by themselves, the unsteady, pressure, viscous, and homogeneous terms of the convective term sum to zero, leaving only the heterogeneous convective terms:

$$\mathbf{v_1}\cdot\nabla\mathbf{v_2} + \mathbf{v_2}\cdot\nabla\mathbf{v_1} = \mathbf{0}$$

This is the condition that must be satisfied for a linear combination of solutions to the Navier-Stokes equations to also be a solution. This condition is solved trivially if there are no spatial derivatives for either velocity solution, as is the case for the trivial solutions derived in Case 1, Equation 6 of the main paper, but is more complex if either solution has spatial dependence. Expanding these equations gives the three component equations:

$$
\begin{aligned}
v_{1x}\frac{\partial v_{2x}}{\partial x} + v_{1y}\frac{\partial v_{2x}}{\partial y} + v_{1z}\frac{\partial v_{2x}}{\partial z} + v_{2x}\frac{\partial v_{1x}}{\partial x} + v_{2y}\frac{\partial v_{1x}}{\partial y} + v_{2z}\frac{\partial v_{1x}}{\partial z} = 0 \\
v_{1x}\frac{\partial v_{2y}}{\partial x} + v_{1y}\frac{\partial v_{2y}}{\partial y} + v_{1z}\frac{\partial v_{2y}}{\partial z} + v_{2x}\frac{\partial v_{1y}}{\partial x} + v_{2y}\frac{\partial v_{1y}}{\partial y} + v_{2z}\frac{\partial v_{1y}}{\partial z} = 0 \\
v_{1x}\frac{\partial v_{2z}}{\partial x} + v_{1y}\frac{\partial v_{2z}}{\partial y} + v_{1z}\frac{\partial v_{2z}}{\partial z} + v_{2x}\frac{\partial v_{1z}}{\partial x} + v_{2y}\frac{\partial v_{1z}}{\partial y} + v_{2z}\frac{\partial v_{1z}}{\partial z} = 0
\end{aligned}
$$

If $\mathbf{v_1}$ and $\mathbf{v_2}$ are solutions as presented in the main paper, in which $\mathbf{v_1}\cdot\nabla\mathbf{v_1} = \mathbf{0}$ and $\mathbf{v_2}\cdot\nabla\mathbf{v_2} = \mathbf{0}$, these equations are solved if the two solutions are parallel, i.e. $\mathbf{v_1} \times \mathbf{v_2} = \mathbf{0}$. This cross product gives the component identities:

$$
\begin{aligned}
v_{1y}v_{2z} &= v_{1z}v_{2y} \\
v_{1x}v_{2z} &= v_{1z}v_{2x} \\
v_{1x}v_{2y} &= v_{1y}v_{2x}
\end{aligned}
$$

Substituting these identities into the above equations gives:

$$
\begin{aligned}
\frac{v_{1z}}{v_{2z}}v_{2x}\frac{\partial v_{2x}}{\partial x} + \frac{v_{1z}}{v_{2z}}v_{2y}\frac{\partial v_{2x}}{\partial y} + \frac{v_{1z}}{v_{2z}}v_{2z}\frac{\partial v_{2x}}{\partial z} + \frac{v_{2z}}{v_{1z}}v_{1x}\frac{\partial v_{1x}}{\partial x} + \frac{v_{2z}}{v_{1z}}v_{1y}\frac{\partial v_{1x}}{\partial y} + \frac{v_{2z}}{v_{1z}}v_{1z}\frac{\partial v_{1x}}{\partial z} = 0 \\
\frac{v_{1z}}{v_{2z}}v_{2x}\frac{\partial v_{2y}}{\partial x} + \frac{v_{1z}}{v_{2z}}v_{2y}\frac{\partial v_{2y}}{\partial y} + \frac{v_{1z}}{v_{2z}}v_{2z}\frac{\partial v_{2y}}{\partial z} + \frac{v_{2z}}{v_{1z}}v_{1x}\frac{\partial v_{1y}}{\partial x} + \frac{v_{2z}}{v_{1z}}v_{1y}\frac{\partial v_{1y}}{\partial y} + \frac{v_{2z}}{v_{1z}}v_{1z}\frac{\partial v_{1y}}{\partial z} = 0 \\
\frac{v_{1z}}{v_{2z}}v_{2x}\frac{\partial v_{2z}}{\partial x} + \frac{v_{1z}}{v_{2z}}v_{2y}\frac{\partial v_{2z}}{\partial y} + \frac{v_{1z}}{v_{2z}}v_{2z}\frac{\partial v_{2z}}{\partial z} + \frac{v_{2z}}{v_{1z}}v_{1x}\frac{\partial v_{1z}}{\partial x} + \frac{v_{2z}}{v_{1z}}v_{1y}\frac{\partial v_{1z}}{\partial y} + \frac{v_{2z}}{v_{1z}}v_{1z}\frac{\partial v_{1z}}{\partial z} = 0
\end{aligned}
$$

which, condensed back to vector notation, become:

$$
\frac{v_{1z}}{v_{2z}}\mathbf{v_2}\cdot\nabla\mathbf{v_2} + \frac{v_{2z}}{v_{1z}}\mathbf{v_1}\cdot\nabla\mathbf{v_1} = \mathbf{0}
$$

which, since each of the convective terms are zero vectors, holds true.

This result indicates that a solution may be generated by combining solutions with multiple length scales λ along the same direction. This superposition of length scales allows the solution method to have greater flexibility in dealing with complex flow systems.

Addendum 2: Additional rotational solutions

In deriving the uniaxial flow solutions found in Equations 14a-14c of the main article, the rotation matrix:

$$\mathbf{R} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

was used to rotate the symmetry-derived solution in Equation 7 of the main paper to coincide with the x-axis. As was mentioned in the article, this matrix is not the only valid rotation matrix for arriving at a uniaxial solution but was instead chosen arbitrarily as an example in order to demonstrate the technique. There are, in fact, an infinite number of valid rotations corresponding to all possible orientations of the y'' and z'' coordinate axes, with respect to the original coordinate axes, within the new $y'' - z''$ plane. These reorientations may be performed after the initial rotation, by applying a second rotational matrix which rotates the reference plane about the x''-axis. For a general rotation of angle θ from the original solution, this rotation is performed by substituting the equation:

$$\begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

into Equations 14a-14c of the main article. Since the x-axis is not rotated and v''_y and v''_z are zero, in this special case the velocity vector does not need to be rotated. This yields a new equation for v''_x:

$$\begin{aligned} v''_x(x^*, y^*, z^*, t) =& Ae^{-2\frac{\eta}{\rho}\lambda^2 t}\Bigg[\cos\left(\lambda\sqrt{2}(y^*\cos\theta + z^*\sin\theta)\right) \\ &+ 2\cos\left(\frac{\lambda}{2}\sqrt{6}(z^*\cos\theta - y^*\sin\theta)\right)\cos\left(\frac{\lambda}{2}\sqrt{2}(y^*\cos\theta + z^*\sin\theta)\right)\Bigg] \\ &+ \frac{1}{\rho}\int \Delta P(t)\partial t + B \end{aligned}$$

where here, the constants have been condensed without loss of generality, and the trigonometric sum-to-product relationship has been used to condense the last two spatially dependent terms.

As shown in Addendum 1, linear combinations of parallel solutions are also solutions, and a separate solution exists for each increment of θ. Therefore, a more general representation of the solution set would be an integral over all incremental angular changes of θ, with appropriate weighting of the function and choice of constants for each angle. Employing this method, the general solution becomes:

$$
\begin{aligned}
v_x(x,y,z,t) = &\sum_i \int_0^{2\pi} \omega_i(\theta) e^{-2\frac{\eta}{\rho}\lambda_i^2 t} \Bigg[\cos\left(\lambda_i \sqrt{2}(y\cos\theta + z\sin\theta)\right) \\
&+ 2\cos\left(\frac{\lambda_i}{2}\sqrt{6}(z\cos\theta - y\sin\theta)\right) + \cos\left(\frac{\lambda_i}{2}\sqrt{2}(y\cos\theta + z\sin\theta)\right)\Bigg]\partial\theta \\
&+ \frac{1}{\rho}\int \Delta P(t)\partial t + B
\end{aligned}
$$

where the integration constant A has been absorbed into the weighting function $\omega_i(\theta)$, and λ has been replaced by λ_i. The summation over i accounts for the superposition of multiple length scales for the same angle of rotation, or non-identical length scales at different angles of rotation. The pressure terms and final integration constants have been combined into single terms without loss of generality, and all superscripts on the velocity, position, pressure and final integration terms have been dropped for clarity, since the solutions remain solutions regardless of the choice of axis system. The velocity terms v_y and v_z remain zero functions. These solutions represent the general rotation case from the $(1\ 1\ 1)$ direction solution.

Addendum 3: Additional steady-state terms

The equations presented in the main paper decay over time to spatially-independent, simple functions of the pressure, which limits their utility for describing complex flow systems, such as sustained turbulent flow. It is therefore worthwhile to examine the possible existence of additional steady-state terms that can help to describe flows of higher complexity.

It is important to note that while additional steady-state solutions may exist, solutions for *arbitrary* initial or final velocity fields will not be found as long as the assumed condition of incompressibility remains valid. All velocity fields found under these conditions must satisfy the incompressible continuity equation, which restricts the possible solutions.

Additional steady-state terms may also be found through linear combinations. Suppose a trial uniaxial solution is given by:

$$
\begin{aligned}
v_x(x,y,z,t) =& w_0(x,y,z) + w(x,y,z,t) \\
v_y(x,y,z,t) =& 0 \\
v_z(x,y,z,t) =& 0
\end{aligned}
$$

where $w(x,y,z,t)$ is a uniaxial solution as given by Equation 14a of the main paper, and $w_0(x,y,z)$ is an unknown steady-state velocity function. Immediately plugging this trial solution into Equation 1b of the main paper indicates that w_0 must be independent of x to satisfy continuity. Plugging the solution with this result into Equation 1a of the main paper gives a single non-trivial component equation:

$$
\rho \frac{\partial w}{\partial t} = -\frac{\partial P}{\partial x} + \eta \left(\frac{\partial^2 (w + w_0)}{\partial y^2} + \frac{\partial^2 (w + w_0)}{\partial z^2} \right)
$$

with the convective terms cancelling to zero because of the value of v_y and v_z and the independence of v_x on x. Because w is a known solution to the Navier-Stokes equation, the terms containing w cancel along with the pressure term, leaving:

$$
\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial^2 w_0}{\partial z^2} = 0
$$

This equation is solved easily by separation of variables, giving:

$$
w_0 = (C \cosh(\gamma y) + D \sinh(\gamma y)) \, (F \cos(\gamma z) + G \cos(\gamma z))
$$

Where C, D, F, and G are integration constants with units of velocity, and γ is a complex constant with units of inverse length. This solution is rotatable about the x-axis, in the same way described for the solution in Addendum 2, and length scales γ are also superimposable as described in Addendum 1. Combining this result with the result from Addendum 2 gives the general uniaxial solution:

$$
\begin{aligned}
v_x(x,y,z,t) =& \sum_i \int_0^{2\pi} \omega_i(\theta) e^{-2\frac{\eta}{\rho}\lambda_i^2 t} \Bigg[\cos\left(\lambda_i \sqrt{2}(y\cos\theta + z\sin\theta)\right) \\
& + 2\cos\left(\frac{\lambda_i}{2}\sqrt{6}(z\cos\theta - y\sin\theta)\right) + \cos\left(\frac{\lambda_i}{2}\sqrt{2}(y\cos\theta + z\sin\theta)\right)\Bigg] \\
& + \omega_i'(\theta)\left[C\cosh\left(\gamma_i(y\cos\theta + z\sin\theta)\right) + D\sinh\left(\gamma_i(y\cos\theta + z\sin\theta)\right)\right] \\
& * \left[F\cos\left(\gamma_i(z\cos\theta - y\sin\theta)\right) + G\cos\left(\gamma_i(z\cos\theta - y\sin\theta)\right)\right]\partial\theta \\
& + \frac{1}{\rho}\int \Delta P(t)\partial t + B \\
v_y(x,y,z,t) =& 0 \\
v_z(x,y,z,t) =& 0
\end{aligned}
$$

where $\omega'(\theta)$ is an additional θ-dependent weighting function for the new linear combination and the superposition of length scales is given by the sum over γ_i. The coordinate axes may then be rotated as described in the main paper to generate solutions for other directions.